Pitanje 14: Osobine diskretnih linearnih vremenski invarijantnih (LTI) sistema

Po analogiji sa kontinualnim sistemima, cilj nam je da se kroz ovo pitanje analiziraju svojstva impulsnih odziva koji zadovoljavaju ili ne zadovoljavaju osobine kao što su kauzalnost, posedovanje memorije, stabilnost i invertibilnost.

Sistemi sa memorijom

Linearni, vremenski invarijantan diskretni sistem bez memorije je definisan relacijom

$$y[n] = Kx[n] \tag{14.1}$$

pa je samim tim njegov impulsni odziv

$$h[n] = K\delta[n] \tag{14.2}$$

Zaključujemo da ako impulsni odziv nekog sistema zadovoljava uslov da je $h[n_0] \neq 0$ za neko $n_0 \neq 0$, da je u pitanju sistem sa memorijom.

Kauzalni sistem

Kauzalnost podrazumeva da sistem ne može da generiše odgovor, odnosno odziv, pre nego što se pobuda pojavi na njegovom ulazu. Dakle, impulsni odziv $h[n]$ nekog kauzalnog sistema mora da zadovolji uslov:

$$h[n] = 0 \quad za \quad n < 0 \tag{14.3}$$

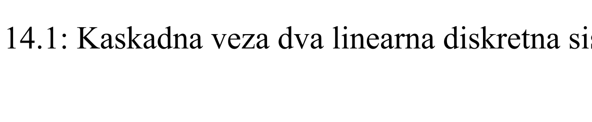
U slučaju kauzalnih sistema, konvolucija kojom se sračunava odziv sistema $y[n]$ na proizvoljnu pobudu $x[n]$ može da se napiše u pojednostavljenoj formi:

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k] = \sum_{l=-\infty}^{n} x[l]h[n-l] \tag{4.4}$$

Iz poslednje jednakosti se jednoznačno vidi da na vrednost odziva $y[n]$ utiču isključivo odbirci signala $x[l]$ za $l \leq n$.

Kaskadna veza

Pod kaskadnom vezom dva diskretna signala podrazumevamo takvu vezu u kojoj je izlaz prvog sistema istovremeno ulaz drugog, kako je to prikazano na slici 14.1.



Slika 14.1: Kaskadna veza dva linearna diskretna sistema

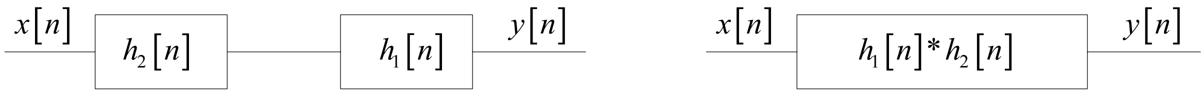
Na osnovu slike 14.1. odziv $y[n]$ možemo napisati u sledećoj formi:

$$y[n] = w[n] * h_2[n] = \left[x[n] * h_1[n]\right] * h_2[n] \tag{14.5}$$

Uzimajući u obzir osobine komutativnosti i asocijativnost operacije diskretne konvolucije, poslednji izraz se može napisati i u sledećoj formi:

$$y[n] = x[n] * \left[h_1[n] * h_2[n]\right] = \left[x[n] * h_2[n]\right] * h_1[n] \tag{14.6}$$

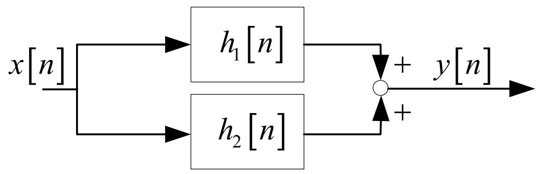
što znači da u kaskadnoj vezi LTI sistemi mogu menjati svoja mesta. Drugi zaključak je da se kaskadna veza dva LTI sistema može ekvivalentno predstaviti jednim LTI sistemom koji ima impulsni odziv jednak konvoluciji impulsnih odziva početnih sistema u kaskadi. Ove dve ekvivalentne strukture su prikazane na slici 14.2.



Slika 14.2: Strukture ekvivalentne kaskadnoj vezi sa slike 5.6

Paralelna veza dva LTI sistema

Pod paralelnom vezom dva LTI sistema se podrazumeva veza dva sistema koji imaju zajednički ulazni signal i pri čemu se njihovi odzivi sabiraju i formiraju zajednički izlaz, kako je to prikazano na slici 14.3.



Slika 14.3: Paralelna veza dva LTI sistema

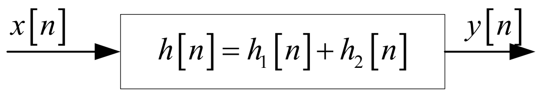
Na osnovu slike 14.3, možemo izračunati odziv sistema $y[n]$ na sledeći način:

$$y[n] = x[n] * h_1[n] + x[n] * h_2[n] \tag{14.7}$$

Uzimajući u obzir osobinu distributivnosti operacije konvolucije nad sabiranjem, poslednji izraz se može napisati na sledeći način:

$$y[n] = x[n] * \left[h_1[n] + h_2[n]\right] \tag{14.8}$$

što nam govori da se paralelna veza dva LTI sistema ekvivalentno može predstaviti kao jedan LTI sistem čiji je impulsni odziv jednak zbiru impulsnih odziva sistema u paraleltnim granama (slika 14.4).



Slika 14.4: Sistem ekvivalentan paralelnoj vezi dva LTI sistema

Stabilnost sistema

BIBO stabilnost diskretnog LTI sistema se vrlo jednostavno može proveriti pomoću impulsnog odziva $h[n]$. Pretpostavimo da je ulazni signal $x[n]$ takav da zadovoljava sledeći uslov:

$$(\forall n)\ |x[n]| \le B_1 \tag{14.9}$$

Tada se apsolutna vrednost odziva $y[n]$ može napisati u sledećoj formi:

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right| \tag{14.10}$$

Znajući da je apsolutna vrednost zbira uvek manja ili jednaka od zbira apsolutnih vrednosti sabiraka, dobijamo sledeću nejednakost:

$$|y[n]| \le \sum_{k=-\infty}^{\infty} |x[k]h[n-k]| = \sum_{k=-\infty}^{\infty} |x[k]||h[n-k]| \le B_1 \sum_{k=-\infty}^{\infty} |h[n-k]| \tag{14.11}$$

Uslov da naš sistem bude BIBO stabilan jeste da beskonačna suma u izrazu (14.11) bude ograničena, odnosno

$$(\exists G) \sum_{k=-\infty}^{\infty} |h[k]| \le G \tag{14.12}$$

U tom slučaju je i apsolutna vrednost odziva ograničena:

$$|y[n]| \le B_1 G = B_2 \tag{14.13}$$

Drugim rečima, potreban i dovoljan uslov da LTI sistem bude BIBO stabilan jeste da njegov impulsni odziv bude apsolutno sumabilan, odnosno da bude zadovoljena relacija (14.12).

Primer 5.3: Impulsni odziv jednog diskretnog LTI sistema glasi:

$$h[n] = a^n u[n] \tag{14.14}$$

Lako možemo proveriti pod kojim uslovima, odnosno za kakvo a je ovaj sistem BIBO stabilan. Ako, shodno relaciji (14.12) sračunamo sumu

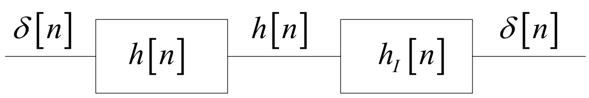
$$\sum_{k=-\infty}^{\infty} |h[k]| = \sum_{k=-\infty}^{\infty} |a^k u[k]| = \sum_{k=0}^{\infty} |a|^k = \lim_{k\to\infty} \frac{1-|a|^k}{1-|a|} \tag{14.15}$$

Očigledno je da poslednja granična vrednost postoji, samo pod uslovom da je $|a|<1$ i tada ona iznosi

$$\sum_{k=-\infty}^{\infty} |h[k]| = \frac{1}{1-|a|}\ ;\ |a|<1 \tag{14.16}$$

Invertibilnost sistema

Pod pretpostavkom da LTI sistem čiji je impulsni odziv $h[n]$ ima inverzni sistem, možemo formirati kaskadnu konekciju originalnog sistema i inverznog sistema kakva je prikazana na slici 14.5.



Slika 14.5: Kaskadna veza originalnog i inverznog sistema

Pod tom pretpostavkom, ako na ulaz originalnog sistema dovedemo diskretni impuls, na njegovom izlazu će se pojaviti impulsni odziv $h[n]$ a na izlazu iznverznog sistema, čiji je impulsni odziv označen sa $h_I[n]$ će se pojaviti ponovo diskretni impuls. Tada možemo pisati:

$$h[n]*h_I[n]=\sum_{k=-\infty}^{\infty}h[k]h_I[n-k]=\delta[n] \tag{14.17}$$

Drugim rečima, da bi LTI diskretni sistem bio invertibilan, potrebno je da postoji signal $h_I[n]$ koji zadovoljava relaciju (14.17) i to će biti impulsni odziv njegovog inverznog sistema. Postupak za nalaženje impulsnog odziva inverznog sistema, ako on postoji, može biti vrlo složen, međutim i ukoliko on postoji i mi ga izračunamo, vrlo često taj inverzni sistem nema neke od nama važnih osobina kao što su kauzalnost i stabilnost.

Pitanje 15: Diferencne jednačine i njihova primena

Uloga diferencnih jednačina u domenu diskretnih sistema je potpuno analogna ulozi diferencijalnih jednačina u prostoru kontinualnih sistema. Opšta forma linearne diferencne jednačine *N*-tog reda sa konstantnim koeficijentima jeste

$$\sum_{k=0}^{N}a_k y[n-k]=\sum_{k=0}^{M}b_k x[n-k] \tag{15.1}$$

Opšte rešenje diferencijalne jednačine se uvek može napisati u obliku:

$$y[n]=y_p[n]+y_h[n] \tag{15.2}$$

gde je sa $y_p[n]$ označeno partikularno rešenje koje zadovoljava jednačinu (15.1), a sa $y_h[n]$ je označeno homogeno rešenje koje zadovoljava diferencnu jednačinu

$$\sum_{k=0}^{N}a_k y[n-k]=0 \tag{15.3}$$

pri čemu se egzaktna forma homogenog rešenja dobija na osnovu dodatnih početnih uslova signala $y[n]$.

Primer 15.1: Posmatrajmo diferencnu jednačinu prvog reda

$$y[n]-ay[n-1]=x[n] \tag{15.4}$$

gde je $x[n]$ kauzalni signal

$$x[n]=b^n u[n] \tag{15.5}$$

Ponovo ćemo rešenje diferencne jednačine tražiti posebno za $n\geq 0$ i za $n<0$. Pretpostavimo da je partikularno rešenje $y_p[n]$ za $n\geq 0$ u obliku:

$$y_p[n]=Ab^n \tag{15.6}$$

Smenom u relaciju (15.4) dobijamo:

$$Ab^n-aAb^{n-1}=b^n \tag{15.7}$$

odnosno

$$Ab - Aa = b \Rightarrow A = \frac{b}{b-a} \tag{15.8}$$

Dakle, partikularno rešenje za nenegativno *n* glasi:

$$y_p[n] = \frac{b^{n+1}}{b-a} \;\; ; \; n \geq 0 \tag{15.9}$$

Homogeno rešenje $y_h[n]$ treba da zadovolji relaciju:

$$y_h[n] - a y_h[n-1] = 0 \tag{15.10}$$

Usvojmo ovo rešenje u obliku:

$$y_h[n] = Kc^n \tag{15.11}$$

gde posle zamene u (15.10) dobijamo:

$$Kc^n - aKc^{n-1} = 0 \Rightarrow Kc - aK = 0 \Rightarrow c = a \tag{15.12}$$

Dakle, homogeno rešenje glasi:

$$y_h[n] = Ka^n \tag{15.13}$$

Kombinujući partikularno i homogeno rešenje za nenegativno *n*, dobijamo

$$y[n] = \frac{b^{n+1}}{b-a} + Ka^n \;\; ; \; n \geq 0 \tag{15.14}$$

Da bismo odredili nepoznatu konstantu *K* potreban nam je jedan početni uslov, recimo $y[-1] = Y_i$. Tada možemo pisati:

$$y[0] - ay[-1] = 1 \Rightarrow y[0] = aY_I + 1 = \frac{b}{b-a} + K \tag{15.15}$$

odnosno

$$K = aY_i - \frac{a}{b-a} \;\; ; \; n \geq 0 \tag{15.16}$$

Konačno rešenje za nenegativno *n* postaje

$$y[n] = Y_i a^{n+1} + \frac{b^{n+1} - a^{n+1}}{b-a} \;\; , \; n \geq 0 \tag{15.17}$$

Za $n < 0$ originalna diferencna jednačina postaje homogena jer je $x[n] = 0$. Dakle, za $n < 0$ rešenje diferencne jednačine je

$$y[n] = y_h[n] = Ka^n \;\; , \;\; n < 0 \tag{15.18}$$

gde se vrednost nepoznatog parametra *K* ponovo određuje iz postojećeg početnog uslova $y[-1] = Y_i$, dakle

$$Ka^{-1} = Y_i \Rightarrow K = Y_i a \;\; ; \; n < 0 \tag{15.19}$$

pa konačno rešenje za negativno *n* postaje:

$$y[n] = Y_i a^{n+1} \;\; ; \; n < 0 \tag{15.20}$$

Rešenja (15.17) i (15.20) se mogu objediniti u jedan zapis na sledeći način:

$$y[n]=Y_i a^{n+1}+\frac{b^{n+1}-a^{n+1}}{b-a}u[n] \tag{15.21}$$

Posmatrajući odziv (15.21) dalje možemo zaključiti da prvi sabirak potiče od prisustva početnog uslova, dok drugi sabirak potiče od prisustva kauzalnog signala $x[n]$. Dakle, rešenje diferencne jednačine se može napisati u obliku:

$$y[n]=y_{zi}[n]+y_{zs}[n] \tag{15.22}$$

Prvi sabirak nosi indeks *zi* (*zero-input*) jer označava da se on dobija kada postoji početni uslov a ne postoji ulaz u sistem $x[n]$ i jednak je:

$$y_{zi}[n]=Y_i a^{n+1} \tag{15.23}$$

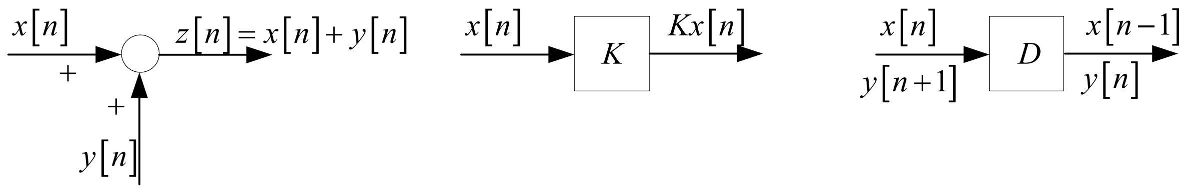
dok drugi sabirak nosi indeks *zs* (*zero states*) jer on označava deo odziva koji potiče od spoljašnjeg signala a ne od početnog uslova:

$$y_{zs}[n]=\frac{b^{n+1}-a^{n+1}}{b-a}u[n] \tag{15.24}$$

Ponovo, ne treba mešati ove sabirke sa partikularnim i homogenim delom rešenja diferencne jednačine jer oni generalno govoreći nisu jednaki.

Blok dijagrami

Slično kao kod kontinualnih sistema, i diskretni sistemi se vrlo često predstavljaju pomoću blok dijagrama gde su elementarni blokovi koji se koriste: sabirači, elementi za kašnjenje (koji se obično obeležavaju sa *D* (*delay*)) i pojačavači ili množači. Ovi blokovi su prikazani na slici 15.1.



Slika 15.1: Elementarni blokovi u diskretnim blok dijagramima (sabirač, množač i element za kašnjenje)

Predstava sistema u obliku blok dijagrama je izuzetno korisna, ne samo sa aspekta lakšeg razumevanja načina na koji sistem funkcioniše, takozvane analize sistema, već i sa aspekta projektovanja diskretnih sistema. Na sledećem primeru ćemo ilustrovati dva različita pristupa predstave diskretnog sistema u formi blok dijagrama.

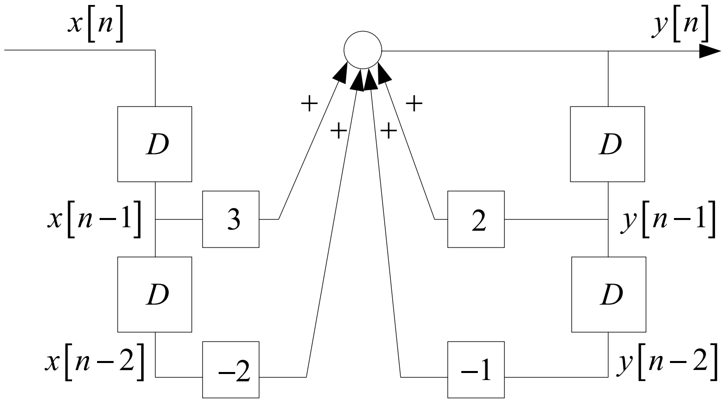
Primer 15.2: Diskretni sistem je opisan diferencnom jednačinom

$$y[n]-2y[n-1]+y[n-2]=3x[n-1]-2x[n-2] \tag{15.25}$$

Kao i u slučaju predstave kontinualnih sistema preko blok dijagrama, najjednostavniji način da se ovaj postupak izvrši jeste takozvana ***direktna realizacija.*** Kao prvi korak, potrebno je izraziti odbirak $y[n]$ preko svih ostalih odbiraka:

$$y[n]=2y[n-1]-y[n-2]+3x[n-1]-2x[n-2] \tag{15.26}$$

i odatle, direktnim postupkom, korišćenjem odgovarajućih blokova za kašnjenje, dobijamo blok dijagram. Ovakav blok dijagram je prikazan na slici 15.2.



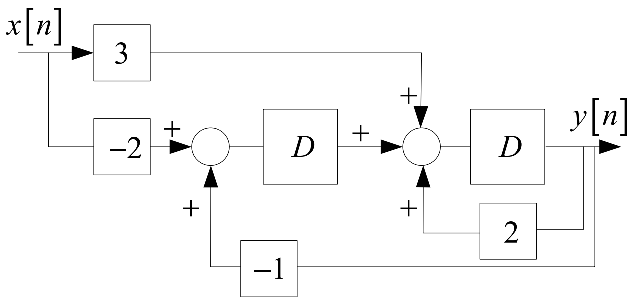
Slika 15.2: Blok dijagram sistema dobijen direktnim postupkom

Iako je direktni postupak vrlo jednostavan u smislu dobijanja blok dijagrama diskretnog sistema, on ima jedan vrlo važan nedostatak, a to je broj elemenata za kašnjenje. Primetimo, sa slike 15.2, da je broj ovih elemenata jednak 4.

Drugi postupak za dobijanje blok strukture sistema jeste takozvana kanonična realizacija. Postupak se sadrži u sledećem: uvedimo oznaku $D\{y[n]\} = y[n-1]$, gde se D tumači kao operator koji primenjen na neki signal vrši njegovo kašnjenje za jedan period odabiranja. Tada relaciju (15.26) možemo napisati na sledeći način:

$$\begin{aligned} y[n] &= 2y[n-1] - y[n-2] + 3x[n-1] - 2x[n-2] \\ &= D\{2y[n] - y[n-1] + 3x[n] - 2x[n-1]\} \\ &= D\{2y[n] + 3x[n] + D\{-y[n] - 2x[n]\}\} \end{aligned} \tag{15.27}$$

Poslednji rezultat se može tumačiti na sledeći način: Signal $y[n]$ se može dobiti kao izlaz iz elementa za jedinično kašnjnje, ako se na ulaz tog elementa dovede zbir tri signala, od kojih je prvi $2y[n]$, drugi $3x[n]$ a treći opet izlaz iz drugog elementa za kašnjenje kome je na ulaz doveden zbir signala $-y[n] - 2x[n]$. Ova ideja je realizovana na slici 15.3. i naziva se kanoničnim blok dijagramom diskretnog sistema.



Slika 5.3: Kanonični blok dijagram diskretnog sistema

Očigledno da je ovakav, kanonični blok dijagram, mnogo povoljniji sa aspekta projektovanja i realizacije sistema s obzirom da je u njega uključeno dva elemenata za kašnjenje, što zapravo predstavlja red diferencne jednačine kojom je sistem definisan u relaciji (15.25).

Furijeova analiza vremenski kontinualnih signala

U dosadašnjim predavanjima smo izvršili analizu svih značajnijih osobina signala do kojih se može doći ukoliko te signale posmatramo u vremenskom domenu. Međutim, već nekoliko vekova unazad razvijene su tehnike transformacije signala koje pružaju značajne a ponekad i pogodnije alate za analizu i sintezu signala i sistema. Vrlo često je priroda nekih signala sa kojima se srećemo svakodnevno, baš takva da je krajnje primerena tim tehnikama. Jedna od tih tehnika jeste transformacija u *frekvencijski domen*. Frekvencijski domen signala nije ništa drugo nego jedan drugačiji pogled na svet oko nas i vrlo često se neki, inače vrlo složeni problemi u vremenskom domenu, vrlo jednostavno rešavaju analizom u frekvencijskom domenu.

Prvi skup takvih transformacija jesu *Furijeova serija* i *Furijeova transformacija* vremenski kontinualnih signala. Ove transformacije su dobile imena po francuskom matematičaru *J.B.J. Fourier*-u koji je postavio teorijske osnove ovih transformacija početkom devetnaestog veka, a polazeći od naučnih radova *Euler*-a iz osamnaestog veka. Osnovni motiv zbog koga se Fourier bavio ovim problemom jeste pokušaj da se opiše propagacija i širenje toplote, dok se Euler bavio analizom vibracije struna žičanih instrumenata. Međutim, dobijeni rezultati su bili u toj meri opšti da su našli primenu u gotovo svim oblastima nauke i tehnike.

Pitanje 16: Sopstvene funkcije kontinualnih LTI sistema

Pretpostavimo da se ulazni ili pobudni signal jednog kontinualnog LTI sistema $x(t)$ može napisati u sledećoj formi:

$$x(t) = \sum_k a_k \phi_k(t) \tag{16.1}$$

gde je skup funkcija $\phi_k(t), k = 0,1,2,...$ na neki način pogodno izabran i naziva se bazisom funkcija, a sa a_k, $k = 0,1,2,...$ su označene odgovarajuće konstante. Znajući da je naš sistem linearan, on zadovoljava osobine homogenosti i aditivnosti, tako da se odziv sistema na ovu pobudu može napisati u sledećoj formi:

$$y(t) = \sum_k a_k \psi_k(t) \tag{16.2}$$

pri čemu je sa $\psi_k(t)$ označen odziv sistema na pobudu $\phi_k(t)$, odnosno, ako sa $h(t)$ označimo impulsni odziv sistema, sledeća relacija je u važnosti:

$$\psi_k(t) = \phi_k(t) * h(t) \tag{16.3}$$

Ovo je vrlo važan rezultat koji se može iskoristiti u velikom broju različitih primera, međutim, njegov značaj postaje još veći ukoliko za bazis funkcija $\phi_k(t)$ izaberemo takve funkcije da su one istog oblika kao i funkcije $\psi_k(t)$ i da se razlikuju samo u jednoj multiplikativnoj konstanti:

$$\psi_k(t) = b_k \phi_k(t) \tag{16.4}$$

jer tada ulaz sistema (16.1) i izlaz sistema (16.2) imaju istu formu. Tada se odziv sistema $y(t)$, umesto relacije (16.2) može napisati u obliku:

$$y(t) = \sum_k c_k \phi_k(t) \tag{16.5}$$

gde je $c_k = a_k b_k$. Takav bazis funkcija $\phi_k(t)$ koji zadovoljava ovu osobinu se naziva bazisom *sopstvenih funkcija* za zadati sistem. U engleskoj literaturi se ove funkcije nazivaju terminom *eigenfunctions*.

Od neprocenjivog značaja je rezultat koji se dobije ukoliko pretpostavimo da su sopstvene funkcije u obliku kompleksnih sinusoida:

$$\phi_k(t) = e^{s_k t} \tag{16.6}$$

sa, za sada, proizvoljnom vrednošću kompleksne konstante s_k. Indeks *k*, takođe, za sada nije strogo definisan i on označava da ovih funkcija u bazisu može biti ili konačan ili beskonačan ali prebrojiv broj. Međutim, pod ovakvim ograničenjem, često nije moguće naći odgovarajući bazis funkcija, pa se zbog toga usvaja da bazis sopstvenih funkcija može da bude skup od neprebrojivo mnogo funkcija. Kaže se da na taj način dozvoljavamo kontinuum sopstvenih funkcija u obliku:

$$\phi(t) = e^{st} \tag{16.7}$$

gde je s kompleksna promenljiva. Lako se pokazuje da funkcija (16.7) zaista predstavlja sopstvenu funkciju LTI sistema. Naime, ako pretpostavimo da je ulazni signal oblika

$$x(t) = \phi(t) = e^{st} \tag{16.8}$$

koristeći konvoluciju, lako sračunavamo odziv sistema

$$\begin{aligned} y(t) &= x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\phi(t-\tau)\,d\tau \\ &= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)}\,d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\,d\tau = H(s)e^{st} \end{aligned} \tag{16.9}$$

gde je sa $H(s)$ označena kompleksna konstanta, koja ne zavisi od vremenske varijable *t*:

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\,d\tau \tag{16.10}$$

Drugim rečima, za pobudu $x(t) = e^{st}$ dobili smo odziv $y(t) = H(s)e^{st}$ za bilo koju vrednost kompleksne promenljive s i bilo koji impulsni odziv $h(t)$, što znači da je $\phi(t) = e^{st}$ sopstvena funkcija proizvoljnog LTI sistema. Ponekada se vrednost $H(s)$ naziva *sopstvenom vrednošću* (u engleskoj literaturi *eigenvalue*) koja odgovara sopstvenoj funkciji $\phi(t)$.

Ovaj rezultat je zapravo osnova transformacija koje ćemo analizirati u ovom i sledećem predavanju, a to su Fourier-ov red, Fourierova transformacija i Laplace-ova transformacija. Specijalno, ako se pobuda $x(t)$ zaista može napisati kao konačna, ili beskonačna ali prebrojiva suma sopstvenih funkcija

$$x(t) = \sum_k a_k \phi_k(t) = \sum_k a_k e^{s_k t} \tag{16.11}$$

tada i odziv sistema dobija istu formu

$$y(t) = \sum_k a_k \psi_k(t) = \sum_k a_k H(s_k) e^{s_k t} \tag{16.12}$$

Fourier-ov red predstavlja periodične signale napisane u ovoj formi (16.12), usvajajući da je $s_k = jk\omega_0$. Ukoliko je priroda signala takva da se ne mogu predstaviti prebrojivom sumom sopstvenih funkcija, već njihovim kontinuumom, opet posežemo za funkcijom tipa $\phi(t) = e^{st}$, a izrazi (16.11) i (16.12) prerastaju u integrale umesto u sume. Ako još usvojimo da je s čisto imaginarni broj $s = j\omega$, dobija se Fourier-ova transformacija. Ako dozvolimo da s bude proizvoljna kompleksna varijabla, dobija se kao rezultat Laplace-ova transformacija. O tome ćemo detaljnije govoriti u ovom i sledećem poglavlju.

Jean Baptiste Joseph Fourier

Born: 21 March 1768 in Auxerre, Bourgogne, France
Died: 16 May 1830 in Paris, France

Joseph Fourier's father was a tailor in Auxerre. After the death of his first wife, with whom he had three children, he remarried and Joseph was the ninth of the twelve children of this second marriage. Joseph's mother died went he was nine years old and his father died the following year.

His first schooling was at Pallais's school, run by the music master from the cathedral. There Joseph studied Latin and French and showed great promise. He proceeded in 1780 to the École Royale Militaire of Auxerre where at first he showed talents for literature but very soon, by the age of thirteen, mathematics became his real interest. By the age of 14 he had completed a study of the six volumes of Bézout's *Cours de mathématiques.* In 1783 he received the first prize for his study of Bossut's *Mécanique en général.*

In 1787 Fourier decided to train for the priesthood and entered the Benedictine abbey of St Benoit-sur-Loire. His interest in mathematics continued, however, and he corresponded with C L Bonard, the professor of mathematics at Auxerre. Fourier was unsure if he was making the right decision in training for the priesthood. He submitted a paper on algebra to Montucla in Paris and his letters to Bonard suggest that he really wanted to make a major impact in mathematics. In one letter Fourier wrote

Yesterday was my 21st birthday, at that age Newton and Pascal had already acquired many claims to immortality.

Fourier did not take his religious vows. Having left St Benoit in 1789, he visited Paris and read a paper on algebraic equations at the Académie Royale des Sciences. In 1790 he became a teacher at the Benedictine college, École Royale Militaire of Auxerre, where he had studied. Up until this time there had been a conflict inside Fourier about whether he should follow a religious life or one of mathematical research. However in 1793 a third element was added to this conflict when he became involved in politics and joined the local Revolutionary Committee. As he wrote:-

As the natural ideas of equality developed it was possible to conceive the sublime hope of establishing among us a free government exempt from kings and priests, and to free from this double yoke the long-usurped soil of Europe. I readily became enamoured of this cause, in my opinion the greatest and most beautiful which any nation has ever undertaken.

Certainly Fourier was unhappy about the Terror which resulted from the French Revolution and he attempted to resign from the committee. However this proved impossible and Fourier was now firmly entangled with the Revolution and unable to withdraw. The revolution was a complicated affair with many factions, with broadly similar aims, violently opposed to each other. Fourier defended members of one faction while in Orléans. A letter describing events relates:-

Citizen Fourier, a young man full of intelligence, eloquence and zeal, was sent to Loiret. ... It seems that Fourier ... got up on certain popular platforms. He can talk very well and if he put forward the views of the Society of Auxerre he has done nothing blameworthy...

This incident was to have serious consequences but after it Fourier returned to Auxerre and continued to work on the revolutionary committee and continued to teach at the College. In July 1794 he was arrested, the charges relating to the Orléans incident, and he was imprisoned. Fourier feared the he would go to the guillotine but, after Robespierre himself went to the guillotine, political changes resulted in Fourier being freed.

Later in 1794 Fourier was nominated to study at the École Normale in Paris. This institution had been set up for training teachers and it was intended to serve as a model for other teacher-training schools. The school opened in January 1795 and Fourier was certainly the most able of the pupils whose abilities ranged widely. He was taught by Lagrange, who Fourier described as

the first among European men of science,

and also by Laplace, who Fourier rated less highly, and by Monge who Fourier described as

having a loud voice and is active, ingenious and very learned.

Fourier began teaching at the Collège de France and, having excellent relations with Lagrange, Laplace and Monge, began further mathematical research. He was appointed to a position at the École Centrale des Travaux Publiques, the school being under the direction of Lazare Carnot and Gaspard Monge, which was soon to be renamed École Polytechnique. However, repercussions of his earlier arrest remained and he was arrested again and imprisoned. His release has been put down to a variety of different causes, pleas by his pupils, pleas by Lagrange, Laplace or Monge or a change in the political climate. In fact all three may have played a part.

By 1 September 1795 Fourier was back teaching at the École Polytechnique. In 1797 he succeeded Lagrange in being appointed to the chair of analysis and mechanics. He was renowned as an outstanding lecturer but he does not appear to have undertaken original research during this time.

In 1798 Fourier joined Napoleon's army in its invasion of Egypt as scientific adviser. Monge and Malus were also part of the expeditionary force. The expedition was at first a great success. Malta was occupied on 10 June 1798, Alexandria taken by storm on 1 July, and the delta of the Nile quickly taken. However, on 1 August 1798 the French fleet was completely destroyed by Nelson's fleet in the Battle of the Nile, so that Napoleon found himself confined to the land that he was occupying. Fourier acted as an administrator as French type political institutions and administration was set up. In particular he helped establish educational facilities in Egypt and carried out archaeological explorations.

While in Cairo Fourier helped found the Cairo Institute and was one of the twelve members of the mathematics division, the others included Monge, Malus and Napoleon himself. Fourier was elected secretary to the Institute, a position he continued to hold during the entire French occupation of Egypt. Fourier was also put in charge of collating the scientific and literary discoveries made during the time in Egypt.

Napoleon abandoned his army and returned to Paris in 1799, he soon held absolute power in France. Fourier returned to France in 1801 with the remains of the expeditionary force and resumed his post as Professor of Analysis at the École Polytechnique. However Napoleon had other ideas about how Fourier might serve him and wrote:-

... the Prefect of the Department of Isère having recently died, I would like to express my confidence in citizen Fourier by appointing him to this place.

Fourier was not happy at the prospect of leaving the academic world and Paris but could not refuse Napoleon's request. He went to Grenoble where his duties as Prefect were many and varied. His two greatest achievements in this administrative position were overseeing the operation to drain the swamps of Bourgoin and supervising the construction of a new highway from Grenoble to Turin. He also spent much time working on the *Description of Egypt* which was not completed until 1810 when Napoleon made changes, rewriting history in places, to it before publication. By the time a second edition appeared every reference to Napoleon would have been removed.

It was during his time in Grenoble that Fourier did his important mathematical work on the theory of heat. His work on the topic began around 1804 and by 1807 he had completed his important memoir *On the Propagation of Heat in Solid Bodies.* The memoir was read to the Paris Institute on 21 December 1807 and a committee consisting of Lagrange, Laplace, Monge and Lacroix was set up to report on the work. Now this memoir is very highly regarded but at the time it caused controversy.

There were two reasons for the committee to feel unhappy with the work. The first objection, made by Lagrange and Laplace in 1808, was to Fourier's expansions of functions as trigonometrical series, what we now call Fourier series. Further clarification by Fourier still failed to convince them. As is pointed out in [4]:-

All these are written with such exemplary clarity - from a logical as opposed to calligraphic point of view - that their inability to persuade Laplace *and* Lagrange *... provides a good index of the originality of Fourier's views.*

The second objection was made by Biot against Fourier's derivation of the equations of transfer of heat. Fourier had not made reference to Biot's 1804 paper on this topic but Biot's paper is certainly incorrect. Laplace, and later Poisson, had similar objections.

The Institute set as a prize competition subject the propagation of heat in solid bodies for the 1811 mathematics prize. Fourier submitted his 1807 memoir together with additional work on the cooling of infinite solids and terrestrial and radiant heat. Only one other entry was received and the committee set up to decide on the award of the prize, Lagrange, Laplace, Malus, Haüy and Legendre, awarded Fourier the prize. The report was not however completely favourable and states:-

... the manner in which the author arrives at these equations is not exempt of difficulties and that his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.

With this rather mixed report there was no move in Paris to publish Fourier's work.

When Napoleon was defeated and on his way to exile in Elba, his route should have been through Grenoble. Fourier managed to avoid this difficult confrontation by sending word that it would be dangerous for Napoleon. When he learnt of Napoleon's escape from Elba and that he was marching towards Grenoble with an army, Fourier was extremely worried. He tried to persuade the people of Grenoble to oppose Napoleon and give their allegiance to the King. However as Napoleon marched into the town Fourier left in haste.

Napoleon was angry with Fourier who he had hoped would welcome his return. Fourier was able to talk his way into favour with both sides and Napoleon made him Prefect of the Rhône. However Fourier soon resigned on receiving orders, possibly from Carnot, that the was to remove all administrators with royalist sympathies. He could not have completely fallen out with Napoleon and Carnot, however, for on 10 June 1815, Napoleon awarded him a pension of 6000 francs, payable from 1 July. However Napoleon was defeated on 1 July and Fourier did not receive any money. He returned to Paris.

Fourier was elected to the Académie des Sciences in 1817. In 1822 Delambre, who was the Secretary to the mathematical section of the Académie des Sciences, died and Fourier together with Biot and Arago applied for the post. After Arago withdrew the election gave Fourier an easy win. Shortly after Fourier became Secretary, the Académie published his prize winning essay *Théorie analytique de la chaleur* in 1822. This was not a piece of political manoeuvring by Fourier however since Delambre had arranged for the printing before he died.

During Fourier's eight last years in Paris he resumed his mathematical researches and published a number of papers, some in pure mathematics while some were on applied mathematical topics. His life was not without problems however since his theory of heat still provoked controversy. Biot claimed priority over Fourier, a claim which Fourier had little difficulty showing to be false. Poisson, however, attacked both Fourier's mathematical techniques and also claimed to have an alternative theory. Fourier wrote *Historical Précis* as a reply to these claims but, although the work was shown to various mathematicians, it was never published.

Fourier's views on the claims of Biot and Poisson are given in the following, see [4]:-

Having contested the various results [Biot *and* Poisson] *now recognise that they are exact but they protest that they have invented another method of expounding them and that this method is excellent and the true one. If they had illuminated this branch of physics by important and general views and had greatly perfected the analysis of partial differential equations, if they had established a principal element of the theory of heat by fine experiments ... they would have the right to judge my work and to correct it. I would submit with much pleasure .. But one does not extend the bounds of science by presenting, in a form said to be different, results which one has not found oneself and, above all, by forestalling the true author in publication.*

Fourier's work provided the impetus for later work on trigonometric series and the theory of functions of a real variable.

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